Observations on the Generating Functions of the Theory of Invariants.

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The source of a covariant is usually understood to be the literal coefficient of the highest power of x; the continuous performance of a certain operation produces the remaining coefficients in succession.

The coefficient of the highest power of y has an equal claim to be considered a source, since the remaining coefficients are derivable by the similar performance of another operation. Moreover, if $f(a_0, a_1, a_2, \ldots)$ be the x-source of a quantic of order ρ, then will the corresponding y-source be

$$f(a_{\rho-0}, a_{\rho-1}, a_{\rho-2}, \ldots).$$

We see that a covariant is a perfectly symmetrical form; it is thus proper to keep both ends of the covariant in view in any investigation of the nature of sources; and, in particular, this is important when laws are being enunciated in regard to the number of sources of any specified type. Consider, as usual, a quantic of order j, and, in connection therewith, algebraic forms of weight $-w$, degree $-i$ and of extent j at most; say these forms are of type (w, i, j). We may discuss x-sources or y-sources of this type, or we may have regard to intermediate coefficients, which Sylvester, in the Phil. Mag. for 1878, called differentiants, so many removes from x or y; thus the coefficient of the highest power but p of x, in a covariant, would be a differentiant, p removes from x, and $\varepsilon - p$ removes from y, where ε is the order of the covariant. In seeking the number of x-sources of type (w, i, j), we have to count the number of partitions of w into i parts none greater than j, and also the number of similar partitions of $w-1$; subtracting the latter number from the former, a number is obtained which, if it be positive, is the number of x-sources of the type in question; but if this difference be negative, it has, as regards x-sources, no interpretation.

To each x-source corresponds a y-source which satisfies the partial differential equation

$$jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots .$$

If the x-source be of type (w, i, j), the connected y-source is of type

$$(w + ji - 2w, i, j) \equiv (ji - w, i, j),$$

and there will be intermediate coefficients $ji - 2w - 1$ in number whenever this number is non-negative; these will be of types

$$\begin{array}{l} (w+1, i, j), \\ (w+2, i, j), \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ (ji - w - 1, i, j), \end{array}$$

respectively.

Had we, *ab initio*, sought the number of y-sources of type $(ji - w, i, j)$, we should have written down the terms of this type, each attached to an arbitrary coefficient, and, by operation of the y-source annihilator, have obtained a set of equations between them in number equal to the number of terms of type $(ji - w + 1, i, j)$. Assuming the non-existence of syzygies connecting these equations, the number of y-sources of the given type would be the difference (when positive), $[ji - w; i, j] - [ji - w + 1, i, j]$, wherein $[w, i, j]$ denotes the number of partitions of type (w, i, j).

We have, as a known theorem of partitions,

$$[wij] - [w-1, i, j] = [ji - w, i, j] - [ji - w + 1, i, j],$$

from which the conclusion is that no syzygies exist between the foregoing set of equations, and that, as a fact, the number of y-sources of type $(w-1, i, j)$ is precisely $[w-1, i, j] - [w; i, j]$, when this number happens to be positive.

If $[w-1, i, j] = [w, i, j]$, there exists neither an x-source of type (w, i, j) nor a y-source of type $(w-1, i, j)$.

There may, however, exist intermediate coefficients of either type, since the dexter sides of the equations arrived at will, in general, not be zero.

We are now in a position to give a more complete statement of the Cayley-Sylvester theorem under discussion.

NEW STATEMENT OF THEOREM.

The number
$$[w, i, j] - [w-1, i, j]$$
is, when positive, the number of x-sources of type (w, i, j); whilst, when
$$[w-1, i, j] - [w, i, j]$$
is positive, it denotes the number of y-sources of type
$$(w-1, i, j).$$
This is equivalent to saying that the coefficient of x^w in the ascending expansion of
$$\frac{1-x^{j+1}, \ldots . 1-x^{j+i}}{1-x^2, \ldots . 1-x^i}$$
is, when positive, the number of x-sources of type (w, i, j); whilst, when in the same expansion the coefficient of $-x^w$ is positive, it denotes the number of y-sources of type $(w-1, i, j)$.

It is somewhat remarkable that this explanation of the complete expansion of the generating function, simple and fundamental as it is, appears to have hitherto escaped the notice of writers on the subject.

It will now be clear that we may write
$$\frac{1-x^{j+1}. 1-x^{j+2}. \ldots . 1-x^{j+i}}{1-x^2. 1-x^3. \ldots . 1-x^i} = \Sigma A_w (x^w - x^{ji+1-w}),$$
A_w being a positive number and denoting the number of x-sources of type (w, i, j).

We may put
$$x = \cosh\phi + \sinh\phi,$$
and then
$$x^s + x^{-s} = 2\cosh s\phi,$$
$$x^s - x^{-s} = 2\sinh s\phi,$$
leading to the identity
$$\frac{\sinh\frac{1}{2}(j+1)\phi \sinh\frac{1}{2}(j+2)\phi \ldots . \sinh\frac{1}{2}(j+i)\phi}{\sinh\phi \sinh\frac{3}{2}\phi \ldots . \sinh\frac{1}{2}i\phi}$$
$$= \sum A_w \sinh\frac{1}{2}(ji+1-2w)\phi;$$
or, writing $\sqrt{-1}\psi$ for ϕ, so that
$$\sinh\phi = \sqrt{-1}\sin\psi,$$

we have also

$$\frac{\sin\frac{1}{2}(j+1)\psi\sin\frac{1}{2}(j+2)\psi\ldots\ldots\sin\frac{1}{2}(j+i)\psi}{\sin\psi\sin\frac{3}{2}\psi\ldots\ldots\sin\frac{1}{2}i\psi}$$

$$=\sum A_w\sin\frac{1}{2}(ji+1-2w)\psi.$$

In a certain sense the left-hand members of these two identities are pure generating functions in the theory, since they are omni-positive in development.

A_w may be represented as the coefficient in a Fourier series, its value as a definite integral being

$$\frac{2}{\pi}\int_0^\pi\frac{\sin\frac{1}{2}(j+1)\psi\ldots\ldots\sin\frac{1}{2}(j+i)\psi}{\sin\psi\ldots\ldots\sin\frac{1}{2}i\psi}\sin\frac{1}{2}(ji+1-2w)\psi d\psi.$$